GENERALIZATIONS OF THE CLASSICAL YANG-BAXTER EQUATION AND $\mathcal{O}$-OPERATORS

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Abstract. Tensor solutions (r-matrices) of the classical Yang-Baxter equation (CYBE) in a Lie algebra, obtained as the classical limit of the R-matrix solution of the quantum Yang-Baxter equation (QYBE), is an important structure appearing in different areas such as integrable systems, symplectic geometry, quantum groups and quantum field theory. Further study of CYBE led to its interpretation as certain operators, giving rise to the concept of $\mathcal{O}$-operators. In [3], the $\mathcal{O}$-operators were in turn interpreted as tensor solutions of CYBE by enlarging the Lie algebra. The purpose of this paper is to extend this study to a more general class of operators that were recently introduced [4] in the study of Lax pairs in integrable systems. Relationship between $\mathcal{O}$-operators, relative differential operators and Rota-Baxter operators are also discussed.

Contents

1. Introduction

This paper studies the relationship of a generalization of $\mathcal{O}$-operators with the classical Yang-Baxter equation (CYBE) [8] and its generalizations.

The CYBE in its original tensor form is the classical limit of the quantum Yang-Baxter equation [7, 21] and has played an important role in integrable systems [1, 2] and Poisson-Lie groups (see [4] and the references therein). However the operator form of CYBE is often more useful [19]. For example, the modified classical Yang-Baxter equation is given in terms of the operator form [19, 9]. This point of view also allowed Kupershmidt [17] to generalize the notion of (operator form of) CYBE to so-called $\mathcal{O}$-operators, which in fact can be traced back to Bordemann [9] in integrable systems.

Key words and phrases. Lie algebra, Yang-Baxter equation, $\mathcal{O}$-operator, Rota-Baxter operator.

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It was shown in [3] that an $\mathcal{O}$-operator on a Lie algebra can be realized as a tensor form solution of CYBE in a larger Lie algebra. Thus these two seemingly distinct approaches to solutions of the classical Yang-Baxter equation are unified.

Since then, both the tensor form approach and the operator form approach of CYBE have been generalized. On one hand, the tensor form of CYBE has been generalized to extended CYBE (ECYBE) and generalized CYBE (GCYBE) with the latter arising naturally from the study of Lie bialgebras [20]. On the other hand, the operator form of CYBE, in its generalized form of $\mathcal{O}$-operators, has been further generalized to extended $\mathcal{O}$-operators with modifications by several parameters. These generalizations have found fruitful applications to Lax pairs, Lie bialgebras and PostLie algebras [4]. These generalizations have also motivated the study of their analogues for associative algebras [5].

The purpose of this paper is to further unify these generalizations of the tensor forms and the operator forms of CYBE in a similar framework as in [3]. We also study the relationship between Rota-Baxter operators and $\mathcal{O}$-operators, and study their differential analogues.

In Section 2, we study the relationship between extended $\mathcal{O}$-operators and extended CYBE. We then establish the relationship between extended $\mathcal{O}$-operators and generalized CYBE in Section 3. Finally in Section 4, we introduce a differential variation of an $\mathcal{O}$-operator, called a relative differential operator. We then show that both an $\mathcal{O}$-operator and a relative differential operator can be regarded as a Rota-Baxter operator on a larger Lie algebra.

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2. Extended $\mathcal{O}$-operators and ECYBE

We first recall in Section 2.1 the definitions of extended $\mathcal{O}$-operators, ECYBE and GCYBE. As a motivation for our study, we also recall their relationship [3] in the special case of $\mathcal{O}$-operators and CYBE. We then establish the relationship between extended $\mathcal{O}$-operators and ECYBE in Section 2.2.

2.1. $\mathcal{O}$-operators and CYBE.

We first recall the classical result that a skew-symmetric solution of CYBE in a Lie algebra gives an $\mathcal{O}$-operator through a duality between tensor product and linear maps. Not every $\mathcal{O}$-operator comes from a solution of CYBE in this way. However, any $\mathcal{O}$-operator can be recovered from a solution of CYBE in a larger Lie algebra.

For the rest of the paper, $\mathbf{k}$ denotes a field whose characteristic is not 2, unless otherwise stated. A Lie algebra is taken to be a Lie algebra over $\mathbf{k}$. A tensor product is also taken over $\mathbf{k}$.

2.1.1. *From CYBE to $\mathcal{O}$-operators.* Let $\mathfrak{g}$ be a Lie algebra. For $r = \sum\limits_i a_i \otimes b_i \in \mathfrak{g}^{\otimes 2}$, we use the notations (in the universal enveloping algebra $U(\mathfrak{g})$):

$$r_{12} = \sum_i a_i \otimes b_i \otimes 1, \quad r_{13} = \sum_i a_i \otimes 1 \otimes b_i, \quad r_{23} = \sum_i 1 \otimes a_i \otimes b_i,$$

and

$$[r_{12}, r_{13}] = \sum_{i,j} [a_i, a_j] \otimes b_i \otimes b_j,\ [r_{13}, r_{23}] = \sum_{i,j} a_i \otimes a_j \otimes [b_i, b_j],\ [r_{12}, r_{23}] = \sum_{i,j} a_i \otimes [b_i, a_j] \otimes b_j\,.$$

If $r \in \mathfrak{g}^{\otimes 2}$ satisfies the **classical Yang-Baxter equation (CYBE)**

$$\mathbf{C}(r) \equiv [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0, \tag{1}$$

then r is called a **solution of CYBE in** $\mathfrak{g}$.

The **twisting operator** $\sigma : \mathfrak{g}^{\otimes 2} \to \mathfrak{g}^{\otimes 2}$ is defined by

$$\sigma(x \otimes y) = y \otimes x, \quad \forall x, y \in \mathfrak{g}.$$

We call $r = \sum\limits_i a_i \otimes b_i \in \mathfrak{g}^{\otimes 2}$ **skew-symmetric** (resp. **symmetric**) if $r = -\sigma(r)$ (resp. $r = \sigma(r)$).

We recall the following classical result [19] that establishes the first connection between CYBE and certain linear operators that had been called **Rota-Baxter operators** [6, 18] in the context of associative algebras which have found applications to renormalization in quantum field theory and number theory [12, 13] recently. Let $\mathfrak{g}$ be a Lie algebra with finite dimension over $\mathbf{k}$. Let

$$\wedge : \mathfrak{g} \otimes \mathfrak{g} \to \mathrm{Hom}(\mathfrak{g}^*, \mathfrak{g}), \quad r \mapsto \hat{r}, \quad \forall r \in \mathfrak{g} \otimes \mathfrak{g},$$

be the usual linear isomorphism, namely, for $r = \sum_i u_i \otimes v_i \in \mathfrak{g}^{\otimes 2}$, we define

$$\hat{r} : \mathfrak{g}^* \to \mathfrak{g}, \quad \hat{r}(a^*) = \sum_i a^*(u_i) v_i, \quad \forall a^* \in \mathfrak{g}^*.$$

In other words,

$$\langle \hat{r}(a^*), b^* \rangle = \langle a^* \otimes b^*, r \rangle, \quad a^*, b^* \in \mathfrak{g}^*,$$

where, for a finite dimensional vector space, $\langle\ ,\ \rangle$ denotes the usual pairings $V \otimes V^* \to \mathbf{k}$ and $V^* \otimes V \to \mathbf{k}$.

Recall that a bilinear form $B(\ ,\ ) : \mathfrak{g} \otimes \mathfrak{g} \to \mathbf{k}$ is called **invariant** if

$$B([x, y], z) = B(x, [y, z]), \quad \forall x, y, z \in \mathfrak{g}.$$

Theorem 2.1. ([19]) *Suppose $\mathfrak{g}$ has a nondegenerate and symmetric bilinear form $B(\ ,\ ) : \mathfrak{g}\otimes\mathfrak{g} \to \mathbf{k}$ which is invariant, allowing us to identify $\mathfrak{g}^*$ with $\mathfrak{g}$. Let $r \in \mathfrak{g}^{\otimes 2}$ be skew-symmetric. Then r is a solution of CYBE if and only if $\hat{r} : \mathfrak{g} \to \mathfrak{g}$ satisfies the* **Rota-Baxter equation** (*of weight 0*)

$$[\hat{r}(x), \hat{r}(y)] = \hat{r}([\hat{r}(x), y] + [x, \hat{r}(y)]), \quad \forall x, y \in \mathfrak{g}. \tag{2}$$

Because of this theorem, Eq. (2) is called the **operator form of CYBE** while Eq. (1) is called the **tensor form of CYBE**. Without assuming the existence of a nondegenerate symmetric invariant bilinear form on $\mathfrak{g}$, it is known that the following result holds ([15]): if the symmetric part $r \in \mathfrak{g} \otimes \mathfrak{g}$ is invariant, then r is a solution of the tensor form of CYBE if and only if $\hat{r} : \mathfrak{g}^* \to \mathfrak{g}$ satisfies

$$[r(a^*), r(b^*)] = r(\mathrm{ad}^*(r(a^*))b^* - \mathrm{ad}^*(r(b^*))a^* + [a^*, b^*]_-), \quad \forall a^*, b^* \in \mathfrak{g}^*, \tag{3}$$

where $[,]_-$ is a Lie bracket on $\mathfrak{g}^*$ defined by

$$[a^*, b^*]_- \equiv -\mathrm{ad}^*((r + r^t)(a^*))b^*, \quad \forall a^*, b^* \in \mathfrak{g}^*, \tag{4}$$

with r^t denoting the transpose of r. When r is skew-symmetric, Eq. (3) becomes

$$[\hat{r}(x), \hat{r}(y)] = \hat{r}(\mathrm{ad}^*\hat{r}(x)(y) - \mathrm{ad}^*\hat{r}(y)(x)), \quad \forall x, y \in \mathfrak{g}^*. \tag{5}$$

which can be regarded as a generalization of the operator form (2) of CYBE.

There is a further generalization [9, 17] of Eq. (5).

Definition 2.2. Let $\mathfrak{g}$ be a Lie algebra. Let $V = (V, \rho)$ be a $\mathfrak{g}$-module, given by a representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ where $\mathfrak{gl}(V)$ is the Lie algebra on $\mathrm{End}(V)$. Denote

$$g \cdot v := \rho(g)(v), \quad \forall g \in \mathfrak{g}, v \in V.$$

A linear map $\alpha : V \to \mathfrak{g}$ is called an $\mathcal{O}$**-operator** if

$$[\alpha(x), \alpha(y)] = \alpha(\alpha(x) \cdot y - \alpha(y) \cdot x), \quad \forall x, y \in V. \tag{6}$$

Thus a skew-symmetric solution $r \in \mathfrak{g}^{\otimes 2}$ of CYBE gives an $\mathcal{O}$-operator $\hat{r} : \mathfrak{g}^* \to \mathfrak{g}$.

2.1.2. *From $\mathcal{O}$-operators to CYBE.* In general it is not true that every $\mathcal{O}$-operator $\alpha : \mathfrak{g}^* \to \mathfrak{g}$ comes from a skew-symmetric solution of CYBE in $\mathfrak{g}$. As we will see next, such an α corresponds to a solution of CYBE in a larger Lie algebra.

The $\mathfrak{g}$-module (V, ρ) defines a Lie algebra bracket $[\ ,\ ]_\rho$ on $\mathfrak{g} \oplus V$, called the **semidirect product** and denoted by $\mathfrak{g} \ltimes_\rho V$, such that

$$[g_1 + v_1, g_2 + v_2]_\rho = [g_1, g_2] + g_1 \cdot v_2 - g_2 \cdot v_1, \quad \forall g_1, g_2 \in \mathfrak{g}, v_1, v_2 \in V. \tag{7}$$

Most of the following results are standard. But we want to use the notations throughout the rest of the paper.

Proposition 2.3. *Let V and W be finite dimensional vector spaces over* $\mathbf{k}$.

(a) *We have the natural isomorphisms*

$$\wedge := \wedge_{V,W} : V \otimes W \cong V^{**} \otimes W \cong \mathrm{Hom}(V^*, W), \quad r \mapsto \hat{r}, \quad \forall r \in V \otimes W, \tag{8}$$

$$\vee := \vee_{\mathrm{Hom}(V,W)} : \mathrm{Hom}(V, W) \cong V^* \otimes W, \quad \alpha \mapsto \check{\alpha}, \quad \forall \alpha \in \mathrm{Hom}(V, W). \tag{9}$$

Thus the maps $\wedge_{V,W}$ and $\vee_{\mathrm{Hom}(V^*,W)}$ are the inverses of each other.

(b) *Define the* **twisting operator** *by*

$$\sigma : V \otimes W \to W \otimes V, \quad v \otimes w \mapsto w \otimes v, \quad \forall v \in V, w \in W. \tag{10}$$

For $\alpha : V^* \to W$, let $\alpha^* : W^* \to V^{**} \cong V$ be the dual map of α.

Then for $r \in V \otimes W$, we have

$$\widehat{\sigma(r)} = \hat{r}^*. \tag{11}$$

(c) *We also have the natural injections*

$$\begin{aligned} T := T_{V \otimes W} : \quad & V \otimes W \to (V \oplus W)^{\otimes 2}, \\ & v \otimes w \mapsto \widetilde{v \otimes w} := (v, 0) \otimes (0, w), \quad \forall v \in V, w \in W. \end{aligned} \tag{12}$$

$$\begin{aligned} T := T_{\mathrm{Hom}(V,W)} : \quad & \mathrm{Hom}(V, W) \to \mathrm{Hom}(V \oplus W^*, V^* \oplus W), \\ & \alpha \mapsto \tilde{\alpha} := \iota_2 \circ \alpha \circ p_1, \quad \forall \alpha \in \mathrm{Hom}(V, W). \end{aligned} \tag{13}$$

Here for vector spaces V_i, $i = 1, 2$, $\iota_i : V_i \to V_1 \oplus V_2$ is the usual inclusion and $p_i : V_1 \oplus V_2 \to V_i$ is the usual projection.

(d) *We have the following commutative diagram.*

$$\begin{tikzcd} \mathrm{Hom}(V, W) \arrow[r, "\vee"] \arrow[d, "T"'] & V^* \otimes W \arrow[d, "T"] \\ \mathrm{Hom}(V \oplus W^*, V^* \oplus W) \arrow[r, "\vee"] & (V^* \oplus W)^{\otimes 2} \end{tikzcd} \tag{14}$$

Proof. We just verify Item (b). Let $\{e_1, ...e_n\}$ be a basis of V and $\{f_1, ..., f_m\}$ be a basis of W. Let $\{e_1^*, ..., e_n^*\}$ and $\{f_1^*, ..., f_m^*\}$ be the corresponding dual bases. Then for $\alpha = \sum_i v_i \otimes w_i \in V \otimes W$, where $v_i \in V, w_i \in W$, $\hat{\alpha}$ is give by $\hat{\alpha}(e_k^*) = \sum_i \langle v_i, e_k^* \rangle w_i$, where $1 \leq k \leq n$. For any $1 \leq s \leq m$, by definition we have

$$\sum_i \langle v_i, e_k^* \rangle \langle w_i, f_s^* \rangle = \langle \hat{\alpha}(e_k^*), f_s^* \rangle = \langle e_k^*, (\hat{\alpha})^*(f_s^*) \rangle.$$

Thus $(\hat{\alpha})^*(f_s^*) = \sum_i v_i \langle w_i, f_s^* \rangle = \widehat{\sigma(\alpha)}(f_s^*)$, as required. □

Let V be a vector space. We also use the following notations:

(15) $$r_\pm = (r \pm \sigma(r))/2, \quad \alpha_\pm := (\alpha \pm \alpha^*)/2, \quad \forall r \in V^{\otimes 2}, \alpha \in \mathrm{Hom}(V^*, V).$$

Note that for any $r \in V \otimes V$, $(\hat{r})_\pm = \widehat{r_\pm}$ by Eq. (11). So the notation $\hat{r}_\pm$ is well-defined.

For a representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ of a Lie algebra $\mathfrak{g}$, let $\rho^* : \mathfrak{g} \to \mathfrak{gl}(V^*)$ be the dual representation. Then $\mathfrak{g} \ltimes_{\rho^*} V^*$ is defined. Suppose that $\mathfrak{g}$ and V are finite dimensional. Then using Proposition 2.3, we have the natural embedding

$$\mathrm{Hom}(V, \mathfrak{g}) \xrightarrow{\vee} V^* \otimes \mathfrak{g} \xrightarrow{T} (\mathfrak{g} \ltimes_{\rho^*} V^*)^{\otimes 2}, \qquad \alpha \longmapsto \check{\alpha} \longmapsto \tilde{\check{\alpha}}\ , \quad \alpha \in \mathrm{Hom}(V, \mathfrak{g}).$$

The following result identifies any $\mathcal{O}$-operator as a solution of CYBE in a suitable Lie algebra.

Theorem 2.4. *([3]) A linear map $\alpha : V \to \mathfrak{g}$ is an $\mathcal{O}$-operator if and only if $\tilde{\check{\alpha}}_- = (\tilde{\check{\alpha}} - \sigma(\tilde{\check{\alpha}}))/2$ is a skew-symmetric solution of CYBE in $\mathfrak{g} \ltimes_{\rho^*} V^*$.*

2.2. **From extended CYBE to extended $\mathcal{O}$-operators.** Recently, the concepts of (the tensor form of) CYBE and $\mathcal{O}$-operators have been generalized, and the connection from CYBE to $\mathcal{O}$-operators has been generalized to this context.

2.2.1. *Extended CYBE.* For any $r = \sum\limits_i a_i \otimes b_i \in \mathfrak{g} \otimes \mathfrak{g}$, we set

$$r_{21} = \sum_i b_i \otimes a_i \otimes 1, \quad r_{32} = \sum_i 1 \otimes b_i \otimes a_i, \quad r_{31} = \sum_i b_i \otimes 1 \otimes a_i.$$

Moreover, we set

$$[(a_1 \otimes a_2 \otimes a_3), (b_1 \otimes b_2 \otimes b_3)] = [a_1, b_1] \otimes [a_2, b_2] \otimes [a_3, b_3], \quad \forall\, a_i, b_i \in \mathfrak{g}, i = 1, 2, 3.$$

Definition 2.5. Let $\mathfrak{g}$ be a Lie algebra. Fix an $\epsilon \in \mathbf{k}$. The equation

(16) $$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = \epsilon[(r_{13} + r_{31}), (r_{23} + r_{32})]$$

is called the **extended classical Yang-Baxter equation of mass** ϵ (or **ECYBE of mass** ϵ in short).

2.2.2. *Extended $\mathcal{O}$-operators.* Our generalization of an $\mathcal{O}$-operator was inspired by two developments. On one hand, since an $\mathcal{O}$-operator is a natural generalization of a Rota-Baxter operator of weight 0, it is desirable to define an $\mathcal{O}$-operator of non-zero weight that generalizes a Rota-Baxter of non-zero weight that was first defined for associative algebras. On the other hand, Semenov-Tian-Shansky [19] introduced the notion of **modified Yang-Baxter equation**

(17) $$[R(x), R(y)] - R([R(x), y] + [x, R(y)]) = -[x, y],$$

where $R : \mathfrak{g} \to \mathfrak{g}$ is a linear operator. Moreover, a **Baxter Lie algebra** was introduced as a Lie algebra with a linear operator R satisfying the modified Yang-Baxter equation [1, 9].

These developments motivated us to give a framework of $\mathcal{O}$-operators with extensions to uniformly treat these generalizations, and to generalize Theorem 2.4 to such extended $\mathcal{O}$-operators.

We first introduce the basic Lie algebra setup.

Definition 2.6. (a) Let $(\mathfrak{g}, [\ ,\ ]_{\mathfrak{g}})$, or simply $\mathfrak{g}$, denote a Lie algebra $\mathfrak{g}$ with Lie bracket $[\ ,\ ]_{\mathfrak{g}}$.

(b) For a Lie algebra $\mathfrak{b}$, let $\mathrm{Der}_{\mathbf{k}}\mathfrak{b}$ denote the Lie algebra of derivations of $\mathfrak{b}$.

(c) Let $\mathfrak{a}$ be a Lie algebra. An $\mathfrak{a}$**-Lie algebra** is a triple $(\mathfrak{b}, [\ ,\ ]_{\mathfrak{b}}, \pi)$ consisting of a Lie algebra $(\mathfrak{b}, [\ ,\ ]_{\mathfrak{b}})$ and a Lie algebra homomorphism $\pi : \mathfrak{a} \to \mathrm{Der}_{\mathbf{k}}\mathfrak{b}$. To simplify the notation, we also let $(\mathfrak{b}, \pi)$ or simply $\mathfrak{b}$ denote $(\mathfrak{b}, [\ ,\ ]_{\mathfrak{b}}, \pi)$.

(d) Let $\mathfrak{a}$ be a Lie algebra and let $(\mathfrak{g}, \pi)$ be an $\mathfrak{a}$-Lie algebra. Let $a \cdot b$ denote $\pi(a)b$ for $a \in \mathfrak{a}$ and $b \in \mathfrak{g}$.

We recall a well-known result in Lie theory which will be used throughout this paper.

Proposition 2.7. [16] *Let $\mathfrak{a}$ be a Lie algebra and let $(\mathfrak{b}, \pi)$ be an $\mathfrak{a}$-Lie algebra. Then there exists a unique Lie algebra structure on the vector space direct sum $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$ retaining the old brackets in $\mathfrak{a}$ and $\mathfrak{b}$ and satisfying $[x, y] = \pi(x)y$ for $x \in \mathfrak{a}$ and $y \in \mathfrak{b}$.*

Remark 2.8. The Lie algebra $\mathfrak{g}$ in the above proposition is called the **semidirect product** of $\mathfrak{a}$ and $\mathfrak{b}$, and we write it as $\mathfrak{g} = \mathfrak{a} \ltimes_{\pi} \mathfrak{b}$. When the Lie bracket in $\mathfrak{b}$ happens to be trivial, i.e., it is a vector space, then we recover the usual semidirect product in Eq. (7).

Definition 2.9. Let $\mathfrak{g}$ be a Lie algebra.

(a) Let $\kappa \in \mathbf{k}$ and let (V, ρ) be a $\mathfrak{g}$-module. A linear map $\beta : V \to \mathfrak{g}$ is called an **antisymmetric** $\mathfrak{g}$**-module homomorphism of mass** κ if

$$\kappa\beta(x) \cdot y + \kappa\beta(y) \cdot x = 0, \tag{18}$$

$$\kappa\beta(\xi \cdot x) = \kappa[\xi, \beta(x)]_{\mathfrak{g}}, \quad \forall x, y \in V, \xi \in \mathfrak{g}. \tag{19}$$

(b) Let $\kappa, \mu \in \mathbf{k}$ and let $(\mathfrak{k}, \pi)$ be a $\mathfrak{g}$-Lie algebra. A linear map $\beta : \mathfrak{k} \to \mathfrak{g}$ is called an **antisymmetric** $\mathfrak{g}$**-module homomorphism of mass** (κ, μ) if β satisfies Eq. (18), Eq. (19) and the following equation:

$$\mu\beta([x, y]_{\mathfrak{k}}) \cdot z = \mu[\beta(x) \cdot y, z]_{\mathfrak{k}}, \quad \forall x, y, z \in \mathfrak{k}. \tag{20}$$

Definition 2.10. Let $\mathfrak{g}$ be a Lie algebra and let $(\mathfrak{k}, \pi)$ be a $\mathfrak{g}$-Lie algebra.

(a) Let $\lambda, \kappa, \mu \in \mathbf{k}$. Fix an antisymmetric $\mathfrak{g}$-module homomorphism $\beta : V \to \mathfrak{g}$ of mass (κ, μ). A linear map $\alpha : \mathfrak{k} \to \mathfrak{g}$ is called an **extended $\mathcal{O}$-operator of weight λ with extension β of mass** (κ, μ) if:

$$[\alpha(x), \alpha(y)]_{\mathfrak{g}} - \alpha(\alpha(x) \cdot y - \alpha(y) \cdot x + \lambda[x, y]_{\mathfrak{k}}) = \kappa[\beta(x), \beta(y)]_{\mathfrak{g}} + \mu\beta([x, y]_{\mathfrak{k}}), \quad \forall x, y \in \mathfrak{k}. \tag{21}$$

(b) We also let (α, β) denote an extended $\mathcal{O}$-operator with extension β.

(c) When (V, ρ) is a $\mathfrak{g}$-module, we regard (V, ρ) as a $\mathfrak{g}$-Lie algebra with the trivial bracket. Then λ, μ are irrelevant. We then call the pair (α, β) **an extended $\mathcal{O}$-operator with extension β of mass** κ.

Definition 2.11. Let $\mathfrak{g}$ be a Lie algebra and $(\mathfrak{k}, \pi)$ be a $\mathfrak{g}$-Lie algebra. Then $\alpha : \mathfrak{k} \to \mathfrak{g}$ is called an $\mathcal{O}$**-operator of weight** $\lambda \in \mathbf{k}$ if it satisfies

$$[\alpha(x), \alpha(y)]_{\mathfrak{g}} = \alpha(\alpha(x) \cdot y - \alpha(y) \cdot x + \lambda[x, y]_{\mathfrak{k}}), \quad \forall x, y \in \mathfrak{k}. \tag{22}$$

When $(\mathfrak{k}, \pi) = (\mathfrak{g}, \mathrm{ad})$, Eq. (22) takes the following form:

$$[\alpha(x), \alpha(y)]_{\mathfrak{g}} = \alpha([\alpha(x), y]_{\mathfrak{g}} + [x, \alpha(y)]_{\mathfrak{g}} + \lambda[x, y]_{\mathfrak{g}}), \quad \forall x, y \in \mathfrak{g}. \tag{23}$$

A linear endomorphism $\alpha : \mathfrak{g} \to \mathfrak{g}$ satisfying Eq. (23) is called a **Rota-Baxter operator of weight** λ [6, 12, 13, 18] (in the Lie algebra context).

2.2.3. *From extended CYBE to extended $\mathcal{O}$-operators.* We next generalize Theorem 2.1.

Lemma 2.12. ([4]) *Let $\mathfrak{g}$ be a Lie algebra with finite* **k***-dimension and $r \in \mathfrak{g} \otimes \mathfrak{g}$ be symmetric. Then the following conditions are equivalent.*

(a) $r \in \mathfrak{g} \otimes \mathfrak{g}$ *is* **invariant***, that is,* $(\mathrm{ad}(x) \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}(x))r = 0, \forall x \in \mathfrak{g}$*;*
(b) $\hat{r} : \mathfrak{g}^* \to \mathfrak{g}$ *is* **antisymmetric***, that is,* $\mathrm{ad}^*(\hat{r}(a^*))b^* + \mathrm{ad}^*(\hat{r}(b^*))a^* = 0, \forall a^*, b^* \in \mathfrak{g}^*$*;*
(c) $\hat{r} : \mathfrak{g}^* \to \mathfrak{g}$ *is* $\mathfrak{g}$**-invariant***, that is,* $\hat{r}(\mathrm{ad}^*(x)a^*) = [x, \hat{r}(a^*)], \forall x \in \mathfrak{g}, a^* \in \mathfrak{g}^*$.

The following result characterizes solutions of ECYBE in a Lie algebra $\mathfrak{g}$ in terms of extended $\mathcal{O}$-operators on $\mathfrak{g}$.

Theorem 2.13. ([4]) *Let $\mathfrak{g}$ be a Lie algebra with finite* **k***-dimension, let $r \in \mathfrak{g}\otimes\mathfrak{g}$ and let $\hat{r} : \mathfrak{g}^* \to \mathfrak{g}$ be the corresponding linear map. Define $\hat{r}_\pm$ by Eq. (15). Suppose that r_+ is invariant. Then r is a solution of ECYBE of mass $\frac{\kappa+1}{4}$:*

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = \frac{\kappa + 1}{4}[(r_{13} + r_{31}), (r_{23} + r_{32})]$$

if and only if $\hat{r}_-$ is an extended $\mathcal{O}$-operator with extention $\hat{r}_+$ of mass κ, i.e., the following equation holds:

$$[\hat{r}_-(a^*), \hat{r}_-(b^*)] - \hat{r}_-(\mathrm{ad}^*(\hat{r}_-(a^*))b^* - \mathrm{ad}^*(\hat{r}_-(b^*))a^*) = \kappa[\hat{r}_+(a^*), \hat{r}_+(b^*)], \quad \forall a^*, b^* \in \mathfrak{g}^*. \tag{24}$$

In the special case when $r_+ = 0$ (hence $\hat{r}_+ = 0$), we obtain Kupershmidt's result Eq. (5) and hence Theorem 2.1.

2.3. **From extended $\mathcal{O}$-operators to ECYBE.** We now start with an arbitrary extended $\mathcal{O}$-operator and characterize it as a solution of ECYBE in a suitable Lie algebra.

Let $\mathfrak{g}$ be a Lie algebra and let (V, ρ) be a $\mathfrak{g}$-module, both with finite **k**-dimensions. Let (V^*, ρ^*) be the dual $\mathfrak{g}$-module and let $\tilde{\mathfrak{g}} = \mathfrak{g} \ltimes_{\rho^*} V^*$. Then from Proposition 2.3, we have the commutative diagram

(25)
$$\begin{tikzcd}
\mathrm{Hom}(V, \mathfrak{g}) \arrow[r, "\vee"] \arrow[d, "T"] & \mathfrak{g} \otimes V^* \arrow[d, "T"] \\
\mathrm{Hom}(\tilde{\mathfrak{g}}^*, \tilde{\mathfrak{g}}) \arrow[r, "\vee"] & \tilde{\mathfrak{g}} \otimes \tilde{\mathfrak{g}}
\end{tikzcd}
\qquad
\begin{tikzcd}
\beta \arrow[r, mapsto] \arrow[d, mapsto] & \check{\beta} \arrow[d, mapsto] \\
\tilde{\beta} \arrow[r, mapsto] & \check{\tilde{\beta}} = \tilde{\check{\beta}}
\end{tikzcd}$$

Lemma 2.14. *Let $\mathfrak{g}$ be a Lie algebra and let (V, ρ) be a $\mathfrak{g}$-module, both with finite* **k***-dimensions. Then $\beta \in \mathrm{Hom}(V, \mathfrak{g})$ is an antisymmetric $\mathfrak{g}$-module homomorphism of mass κ if and only if $\tilde{\beta}_+ \in \mathrm{Hom}(\tilde{\mathfrak{g}}^*, \tilde{\mathfrak{g}})$ is an antisymmetric $\tilde{\mathfrak{g}}$-module homomorphism of mass κ.*

Proof. The case that $\kappa = 0$ is obvious. So we suppose that $\kappa \neq 0$. Then antisymmetric of mass κ is the same as antisymmetric (of mass 1) since we assume that **k** is a field. Note that for any $a^* \in \mathfrak{g}^*$ and $u \in V$, we have $\tilde{\beta}_+(a^*) = \beta^*(a^*)/2$ and $\tilde{\beta}_+(u) = \beta(u)/2$ where $\beta^* : \mathfrak{g}^* \to V^*$ is the dual linear map associated to β. In fact, we have $\tilde{\beta}_+ = (\tilde{\beta} + \tilde{\beta}^*)/2$. Moreover, for any $a^* \in \mathfrak{g}^*, u \in V$,

$$\tilde{\beta}(a^*) = l_2 \circ \beta \circ p_1(a^*) = 0,$$
$$\tilde{\beta}(u) = l_2 \circ \beta \circ p_1(u) = \beta(u).$$

Hence for any $b^* \in \mathfrak{g}^*, v \in V$,

$$\langle \tilde{\beta}^*(a^*), v \rangle = \langle a^*, \tilde{\beta}(v) \rangle = \langle a^*, \beta(v) \rangle = \langle \beta^*(a^*), v \rangle;$$

$$\langle\tilde{\beta}^*(a^*), b^*\rangle = \langle a^*, \tilde{\beta}(b^*)\rangle = 0;$$
$$\langle\tilde{\beta}^*(u), a^*\rangle = \langle u, \tilde{\beta}(a^*)\rangle = 0;$$
$$\langle\tilde{\beta}^*(u), v\rangle = \langle u, \tilde{\beta}(v)\rangle = \langle u, \beta(v)\rangle = 0.$$

So we have $\tilde{\beta}_+(a^*) = (\tilde{\beta}(a^*) + \tilde{\beta}^*(a^*))/2 = \beta^*(a^*)/2$ and $\tilde{\beta}_+(u) = (\tilde{\beta}(u) + \tilde{\beta}^*(u))/2 = \beta(u)/2$.

Now suppose that $\beta : (V,\rho) \to \mathfrak{g}$ is an antisymmetric $\mathfrak{g}$-module homomorphism of mass κ. Let $b^* \in \mathfrak{g}^*, v \in V$, then

$$\mathrm{ad}^*_{\tilde{\mathfrak{g}}}(\tilde{\beta}_+(a^* + u))(b^* + v) = (1/2)(\mathrm{ad}^*_{\tilde{\mathfrak{g}}}(\beta^*(a^*))b^* + \mathrm{ad}^*_{\tilde{\mathfrak{g}}}(\beta^*(a^*))v + \mathrm{ad}^*_{\tilde{\mathfrak{g}}}(\beta(u))b^* + \mathrm{ad}^*_{\tilde{\mathfrak{g}}}(\beta(u))v),$$
$$\mathrm{ad}^*_{\tilde{\mathfrak{g}}}(\tilde{\beta}_+(b^* + v))(a^* + u) = (1/2)(\mathrm{ad}^*_{\tilde{\mathfrak{g}}}(\beta^*(b^*))a^* + \mathrm{ad}^*_{\tilde{\mathfrak{g}}}(\beta^*(b^*))u + \mathrm{ad}^*_{\tilde{\mathfrak{g}}}(\beta(v))a^* + \mathrm{ad}^*_{\tilde{\mathfrak{g}}}(\beta(v))u).$$

On the other hand, for any $x \in \mathfrak{g}, w^* \in V^*$,

$$\langle \mathrm{ad}^*_{\tilde{\mathfrak{g}}}(\beta^*(a^*))b^* + \mathrm{ad}^*_{\tilde{\mathfrak{g}}}(\beta^*(b^*))a^*, x\rangle = \langle b^*, [x, \beta^*(a^*)]\rangle + \langle a^*, [x, \beta^*(b^*)]\rangle = 0,$$
$$\langle \mathrm{ad}^*_{\tilde{\mathfrak{g}}}(\beta^*(a^*))b^* + \mathrm{ad}^*_{\tilde{\mathfrak{g}}}(\beta^*(b^*))a^*, w^*\rangle = \langle b^*, [w^*, \beta^*(a^*)]\rangle + \langle a^*, [w^*, \beta^*(b^*)]\rangle = 0,$$
$$\langle \mathrm{ad}^*_{\tilde{\mathfrak{g}}}(\beta^*(a^*))v + \mathrm{ad}^*_{\tilde{\mathfrak{g}}}(\beta(v))a^*, x\rangle = \langle v, [x, \beta^*(a^*)]\rangle + \langle a^*, [x, \beta(v)]\rangle = -\langle \beta(\rho(x)v), a^*\rangle + \langle a^*, [x, \beta(v)]\rangle = 0,$$
$$\langle \mathrm{ad}^*_{\tilde{\mathfrak{g}}}(\beta^*(a^*))v + \mathrm{ad}^*_{\tilde{\mathfrak{g}}}(\beta(v))a^*, w^*\rangle = \langle v, [w^*, \beta^*(a^*)]\rangle + \langle a^*, [w^*, \beta(v)]\rangle = 0,$$
$$\langle \mathrm{ad}^*_{\tilde{\mathfrak{g}}}(\beta(u))b^* + \mathrm{ad}^*_{\tilde{\mathfrak{g}}}(\beta^*(b^*))u, x\rangle = \langle b^*, [x, \beta(u)]\rangle + \langle u, [x, \beta^*(b^*)]\rangle = \langle b^*, [x, \beta(u)]\rangle - \langle \beta(\rho(x)u), b^*\rangle = 0,$$
$$\langle \mathrm{ad}^*_{\tilde{\mathfrak{g}}}(\beta(u))b^* + \mathrm{ad}^*_{\tilde{\mathfrak{g}}}(\beta^*(b^*))u, w^*\rangle = \langle b^*, [w^*, \beta(u)]\rangle + \langle u, [w^*, \beta^*(b^*)]\rangle = 0,$$
$$\langle \mathrm{ad}^*_{\tilde{\mathfrak{g}}}(\beta(u))v + \mathrm{ad}^*_{\tilde{\mathfrak{g}}}(\beta(v))u, x\rangle = \langle v, [x, \beta(u)]\rangle + \langle u, [x, \beta(v)]\rangle = 0,$$
$$\langle \mathrm{ad}^*_{\tilde{\mathfrak{g}}}(\beta(u))v + \mathrm{ad}^*_{\tilde{\mathfrak{g}}}(\beta(v))u, w^*\rangle = \langle v, [w^*, \beta(u)]\rangle + \langle u, [w^*, \beta(v)]\rangle = \langle \rho(\beta(u))v + \rho(\beta(v))u, w^*\rangle = 0.$$

Therefore, $\mathrm{ad}^*_{\tilde{\mathfrak{g}}}(\tilde{\beta}_+(a^* + u))(b^* + v) + \mathrm{ad}^*_{\tilde{\mathfrak{g}}}(\tilde{\beta}_+(b^* + v))(a^* + u) = 0$. Since $\tilde{\tilde{\beta}}_+ \in \tilde{\mathfrak{g}} \otimes \tilde{\mathfrak{g}}$ is symmetric, by Lemma 2.12, $\tilde{\beta}_+$ is an antisymmetric $\tilde{\mathfrak{g}}$-module homomorphism of mass κ.

Conversely, if $\tilde{\beta}_+$ is an antisymmetric $\tilde{\mathfrak{g}}$-module homomorphism of mass κ, then for any $u, v \in V, x \in \mathfrak{g}$,

$$\mathrm{ad}^*_{\tilde{\mathfrak{g}}}(\tilde{\beta}_+(u))v + \mathrm{ad}^*_{\tilde{\mathfrak{g}}}(\tilde{\beta}_+(v))u = 0 \quad \Leftrightarrow \quad \rho(\beta(u))v + \rho(\beta(v))u = 0,$$
$$\tilde{\beta}_+(\mathrm{ad}^*_{\tilde{\mathfrak{g}}}(x)v) = [x, \tilde{\beta}_+(v)] \quad \Leftrightarrow \quad \beta(\rho(x)v) = [x, \beta(v)].$$

So $\beta : (V,\rho) \to \mathfrak{g}$ is an antisymmetric $\mathfrak{g}$-module homomorphism of mass κ. □

Theorem 2.15. *Let $\mathfrak{g}$ be a Lie algebra and (V,ρ) be a $\mathfrak{g}$-module, both with finite* **k***-dimensions. Let $\alpha, \beta : V \to \mathfrak{g}$ be two linear maps. Using the notations in Eq. (25), α is an extended $\mathcal{O}$-operator with extension β of mass κ if and only if $\tilde{\alpha}_-$ is an extended $\mathcal{O}$-operator with extension $\tilde{\beta}_+$ of mass κ.*

Proof. Note that for any $a^* \in \mathfrak{g}^*, v \in V$, we have $\tilde{\alpha}_-(a^*) = -\alpha^*(a^*)/2$ and $\tilde{\alpha}_-(v) = \alpha(v)/2$ where $\alpha^* : \mathfrak{g}^* \to V^*$ is the dual linear map of α. Suppose that α is an extended $\mathcal{O}$-operator with extension β of mass κ. Then for any $a^*, b^* \in \mathfrak{g}^*, u, v \in V$, we have

$$\begin{aligned}
& [\tilde{\alpha}_-(u + a^*), \tilde{\alpha}_-(v + b^*)] - \tilde{\alpha}_-(\mathrm{ad}^*_{\tilde{\mathfrak{g}}}(\tilde{\alpha}_-(u + a^*))(v + b^*) - \mathrm{ad}^*_{\tilde{\mathfrak{g}}}(\tilde{\alpha}_-(v + b^*))(u + a^*)) \\
= & (1/4)\{[\alpha(u), \alpha(v)] - [\alpha(u), \alpha^*(b^*)] - [\alpha^*(a^*), \alpha(v)] + [\alpha^*(a^*), \alpha^*(b^*)]\} \\
& -(1/2)\tilde{\alpha}_-(\mathrm{ad}^*_{\tilde{\mathfrak{g}}}(\alpha(u))v + \mathrm{ad}^*_{\tilde{\mathfrak{g}}}(\alpha(u))b^* - \mathrm{ad}^*_{\tilde{\mathfrak{g}}}(\alpha^*(a^*))v - \mathrm{ad}^*_{\tilde{\mathfrak{g}}}(\alpha^*(a^*))b^* \\
& -\mathrm{ad}^*_{\tilde{\mathfrak{g}}}(\alpha(v))u - \mathrm{ad}^*_{\tilde{\mathfrak{g}}}(\alpha(v))a^* + \mathrm{ad}^*_{\tilde{\mathfrak{g}}}(\alpha^*(b^*))u + \mathrm{ad}^*_{\tilde{\mathfrak{g}}}(\alpha^*(b^*))a^*) \\
= & (1/4)\{[\alpha(u), \alpha(v)] - \alpha(\rho(\alpha(u))v) + \alpha(\rho(\alpha(v))u) - \rho^*(\alpha(u))\alpha^*(b^*) + \alpha^*(\mathrm{ad}^*(\alpha(u))b^*) \\
& +\alpha^*(\mathrm{ad}^*_{\tilde{\mathfrak{g}}}(\alpha^*(b^*))u) + \rho^*(\alpha(v))\alpha^*(a^*) - \alpha^*(\mathrm{ad}^*_{\tilde{\mathfrak{g}}}(\alpha^*(a^*))v) - \alpha^*(\mathrm{ad}^*(\alpha(v))a^*)\}.
\end{aligned}$$

On the other hand, for any $w \in V$, we have

$$\begin{aligned}
& \langle -\rho^*(\alpha(u))\alpha^*(b^*) + \alpha^*(\mathrm{ad}^*(\alpha(u))b^*) + \alpha^*(\mathrm{ad}^*_{\tilde{\mathfrak{g}}}(\alpha^*(b^*))u), w\rangle \\
= & \langle b^*, \alpha(\rho(\alpha(u))w) + [\alpha(w), \alpha(u)] - \alpha(\rho(\alpha(w))u)\rangle \\
= & \langle b^*, k[\beta(w), \beta(u)]\rangle \\
= & \langle b^*, -k\beta(\rho(\beta(u))w)\rangle \\
= & \langle k\rho^*(\beta(u))\beta^*(b^*), w\rangle.
\end{aligned}$$

Thus

$$-\rho^*(\alpha(u))\alpha^*(b^*) + \alpha^*(\mathrm{ad}^*(\alpha(u))b^*) + \alpha^*(\mathrm{ad}^*_{\tilde{\mathfrak{g}}}(\alpha^*(b^*))u) = k\rho^*(\beta(u))\beta^*(b^*).$$

Similarly,

$$\rho^*(\alpha(v))\alpha^*(a^*) - \alpha^*(\mathrm{ad}^*_{\tilde{\mathfrak{g}}}(\alpha^*(a^*))v) - \alpha^*(\mathrm{ad}^*(\alpha(v))a^*) = -k\rho^*(\beta(v))\beta^*(a^*).$$

So

$$\begin{aligned}
& [\tilde{\alpha}_-(u+a^*), \tilde{\alpha}_-(v+b^*)] - \tilde{\alpha}_-(\mathrm{ad}^*_{\tilde{\mathfrak{g}}}(\tilde{\alpha}_-(u+a^*))(v+b^*) - \mathrm{ad}^*_{\tilde{\mathfrak{g}}}(\tilde{\alpha}_-(v+b^*))(u+a^*)) \\
= & (1/4)(\kappa[\beta(u), \beta(v)] + \kappa\rho^*(\beta(u))\beta^*(b^*) - \kappa\rho^*(\beta(v))\beta^*(a^*)) \\
= & (1/4)(\kappa[\beta(u), \beta(v)] + \kappa[\beta(u), \beta^*(b^*)] + \kappa[\beta^*(a^*), \beta(v)]) \\
= & \kappa[\tilde{\beta}_+(u+a^*), \tilde{\beta}_+(v+b^*)].
\end{aligned}$$

Furthermore, since β is an antisymmetric $\mathfrak{g}$-module homomorphism of mass κ, by Lemma 2.14, the linear map $\tilde{\beta}_+$ from $(\tilde{\mathfrak{g}}^*, \mathrm{ad}^*_{\tilde{\mathfrak{g}}})$ to $\tilde{\mathfrak{g}}$ is an antisymmetric $\tilde{\mathfrak{g}}$-module homomorphism of mass κ. Therefore $\tilde{\alpha}_-$ is an extended $\mathcal{O}$-operator with extension $\tilde{\beta}_+$ of mass κ.

Conversely, if $\tilde{\alpha}_-$ is an extended $\mathcal{O}$-operator with extension β of mass κ. Then the linear map $\tilde{\beta}_+$ from $(\tilde{\mathfrak{g}}^*, \mathrm{ad}^*_{\tilde{\mathfrak{g}}})$ to $\tilde{\mathfrak{g}}$ is an antisymmetric $\tilde{\mathfrak{g}}$-module homomorphism of mass κ, which by Lemma 2.14 implies that β is an antisymmetric $\mathfrak{g}$-module homomorphism of mass κ. Moreover, for any $u, v \in V$, we have that

$$[\tilde{\alpha}_-(u), \tilde{\alpha}_-(v)] - \tilde{\alpha}_-(\mathrm{ad}^*_{\tilde{\mathfrak{g}}}(\tilde{\alpha}_-(u))v - \mathrm{ad}^*_{\tilde{\mathfrak{g}}}(\tilde{\alpha}_-(v))u) = \kappa[\tilde{\beta}_+(u), \tilde{\beta}_+(v)].$$

Hence

$$[\alpha(u), \alpha(v)] - \alpha(\rho(\alpha(u))v - \rho(\alpha(v))u) = \kappa[\beta(u), \beta(v)].$$

So α is an extended $\mathcal{O}$-operator with extension β of mass κ. $\square$

Theorem 2.15 allows us to give the following characterization of extended $\mathcal{O}$-operators in terms of solutions of CYBE in a suitable Lie algebra. In particular, Baxter Lie algebras are described by CYBE [1, 9].

Corollary 2.16. *Let $\mathfrak{g}$ be a Lie algebra and let (V, ρ) be a $\mathfrak{g}$-module, both with finite* **k**-*dimension.*

(a) *Let $\alpha, \beta : V \to \mathfrak{g}$ be linear maps. Then α is an extended $\mathcal{O}$-operator with extention β of mass k if and only if $\tilde{\tilde{\alpha}}_- \pm \tilde{\tilde{\beta}}_+$ is a solution of ECYBE of mass $\frac{\kappa+1}{4}$ in $\mathfrak{g} \ltimes_{\rho^*} V^*$.*

(b) ([3]) *Let $\alpha : V \to \mathfrak{g}$ be a linear map. Then α is an $\mathcal{O}$-operator of weight zero if and only if $\tilde{\tilde{\alpha}}_-$ is a skew-symmetric solution of CYBE in $\mathfrak{g} \ltimes_{\rho^*} V^*$. In particular, a linear map $P : \mathfrak{g} \to \mathfrak{g}$ is a Rota-Baxter operator of weight zero if and only if $r = \tilde{\tilde{P}}_-$ is a skew-symmetric solution of CYBE in $\mathfrak{g} \ltimes_{\mathrm{ad}^*} \mathfrak{g}^*$.*

(c) *Let $R : \mathfrak{g} \to \mathfrak{g}$ be a linear map. Then $(\mathfrak{g}, R)$ is a* **Baxter Lie algebra**, *i.e., the following equation holds:*

$$[R(x), R(y)] - R([R(x), y] + [x, R(y)]) = -[x, y], \quad \forall x, y \in \mathfrak{g}. \tag{26}$$

if and only if $\tilde{\tilde{R}}_- \pm \tilde{\tilde{\mathrm{id}}}_+$ *is a solution of CYBE in* $\mathfrak{g} \ltimes_{\mathrm{ad}^*} \mathfrak{g}^*$.

(d) *Let* $P : \mathfrak{g} \to \mathfrak{g}$ *be a linear map. Then* P *is a Rota-Baxter operator of weight* $\lambda \neq 0$ *if and only if both* $\frac{2}{\lambda}\tilde{\tilde{P}}_- + 2\tilde{\tilde{\mathrm{id}}}$ *and* $\frac{2}{\lambda}\tilde{\tilde{P}}_- - 2\sigma(\tilde{\tilde{\mathrm{id}}})$ *are solutions of CYBE in* $\mathfrak{g} \ltimes_{\mathrm{ad}^*} \mathfrak{g}^*$.

Proof. (a) This follows from Theorem 2.15 and Theorem 2.13.

(b) This follows from Theorem 2.15 for $\kappa = 0$ (or $\beta = 0$) and Eq. (5).

(c) This follows from Item (a) in the case that $(V, \rho) = (\mathfrak{g}, \mathrm{ad})$, $\kappa = -1$ and $\beta = \mathrm{id}$.

(d) By [11], P is a Rota-Baxter operator of weight $\lambda \neq 0$ if and only if $\frac{2P}{\lambda} + \mathrm{id}$ is an extended $\mathcal{O}$-operator with extention id of mass -1 from $(\mathfrak{g}, \mathrm{ad})$ to $\mathfrak{g}$, i.e., $\frac{2P}{\lambda} + \mathrm{id}$ satisfies Eq. (26). Then the conclusion follows from Item (c). □

3. Extended $\mathcal{O}$-operators and generalized CYBE

In this section, we consider the relationship between extended $\mathcal{O}$-operators and the generalized CYBE.

Recall that a **Lie bialgebra** structure on a Lie algebra $\mathfrak{g}$ is a skew-symmetric $\mathbf{k}$-linear map $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$, called the **cocommutator**, such that $(\mathfrak{g}, \delta)$ is a Lie coalgebra and δ is a 1-cocycle of $\mathfrak{g}$ with coefficients in $\mathfrak{g} \otimes \mathfrak{g}$, that is, δ satisfies the following equation:

$$\delta([x, y]) = (\mathrm{ad}(x) \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}(x))\delta(y) - (\mathrm{ad}(y) \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}(y))\delta(x), \quad \forall x, y \in \mathfrak{g}.$$

Proposition 3.1. ([10]) *Let* $\mathfrak{g}$ *be a Lie algebra and* $r \in \mathfrak{g} \otimes \mathfrak{g}$. *Define a linear map* $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ *by*

$$\delta(x) = (\mathrm{ad}(x) \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}(x))r, \quad \forall x \in \mathfrak{g}. \tag{27}$$

Then $(\mathfrak{g}, \delta)$ *becomes a Lie coalgebra, i.e.,* $\delta^* : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ *defines a Lie algebra structure on* $\mathfrak{g}$, *if and only if the following conditions are satisfied for all* $x \in \mathfrak{g}$:

(a) $(\mathrm{ad}(x) \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}(x))r_+ = 0$ *for* r_+ *defined in Eq. (15).*
(b) $(\mathrm{ad}(x) \otimes \mathrm{id} \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}(x) \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{id} \otimes \mathrm{ad}(x))([r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}]) = 0.$

Such a Lie bialgebra $(\mathfrak{g}, \delta)$ is called a **coboundary Lie bialgebra** [10].

Definition 3.2. ([20]) *Let* $\mathfrak{g}$ *be a Lie algebra. The following equation is called the* **generalized classical Yang-Baxter equation (GCYBE)** *in* $\mathfrak{g}$:

$$(\mathrm{ad}(x) \otimes \mathrm{id} \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}(x) \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{id} \otimes \mathrm{ad}(x))([r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}]) = 0, \ \forall x \in \mathfrak{g}. \tag{28}$$

Lemma 3.3. ([4]) *Let* $\mathfrak{g}$ *be a Lie algebra with finite* $\mathbf{k}$*-dimension and let* $r \in \mathfrak{g} \otimes \mathfrak{g}$. *Let* $[,]_\delta$ *be the bracket on* $\mathfrak{g}^*$ *induced by Eq. (27), defined by*

$$\langle [a^*, b^*]_\delta, x \rangle = \langle a^* \otimes b^*, \delta(x) \rangle, \quad \forall x \in \mathfrak{g}, a^*, b^* \in \mathfrak{g}^*.$$

Then for the $\hat{r} : \mathfrak{g}^* \to \mathfrak{g}$ *induced from* r, *we have*

$$[a^*, b^*]_\delta = \mathrm{ad}^*(\hat{r}(a^*))b^* + \mathrm{ad}^*(\hat{r}^*(a^*))b^*, \quad \forall a^*, b^* \in \mathfrak{g}^*. \tag{29}$$

Further, let $\hat{r}_\pm : \mathfrak{g}^* \to \mathfrak{g}$ *be the two linear maps given by Eq. (15). If* r_+ *is invariant, then*

$$[a^*, b^*]_\delta = \mathrm{ad}^*(\hat{r}_-(a^*))b^* - \mathrm{ad}^*(\hat{r}_-(a^*))b^*, \quad \forall a^*, b^* \in \mathfrak{g}^*. \tag{30}$$

By Proposition 3.1 and Lemma 3.3, one can get the following known conclusion:

Corollary 3.4. ([14, 15]) *Let $\mathfrak{g}$ be a Lie algebra and $r \in \mathfrak{g} \otimes \mathfrak{g}$. Suppose that r is skew-symmetric, i.e., $r_+ = 0$. Then r is a solution of GCYBE if and only if Eq. (30) defines a Lie bracket on $\mathfrak{g}^*$.*

Lemma 3.5. *Let $\mathfrak{g}$ be a Lie algebra and $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of $\mathfrak{g}$ on a vector space V. Let $\alpha : V \to \mathfrak{g}$ be a linear map. Then the bracket*

$$[u, v]_\alpha := \rho(\alpha(u))v - \rho(\alpha(v))u, \quad \forall u, v \in V, \tag{31}$$

defines a Lie algebra structure on V if and only if the following equation holds:

$$\rho([\alpha(v), \alpha(u)] - \alpha(\rho(\alpha(v))u - \rho(\alpha(u))v)w + \text{cycl.} = 0, \quad \forall u, v \in V. \tag{32}$$

Here for an expression $f(u, v, w)$ in u, v, w, $f(u, v, w)+$cycl. means $f(u, v, w)+f(v, w, u)+f(w, u, v)$.

Proof. For any $u, v, w \in V$, we have

$$\begin{aligned}
[[u, v]_\alpha, w]_\alpha &= \rho(\alpha(\rho(\alpha(u))v - \rho(\alpha(v))u))w - \rho(\alpha(w))\rho(\alpha(u))v + \rho(\alpha(w))\rho(\alpha(v))u,\\
[[w, u]_\alpha, v]_\alpha &= \rho(\alpha(\rho(\alpha(w))u - \rho(\alpha(u))w))v - \rho(\alpha(v))\rho(\alpha(w))u + \rho(\alpha(v))\rho(\alpha(u))w,\\
[[v, w]_\alpha, u]_\alpha &= \rho(\alpha(\rho(\alpha(v))w - \rho(\alpha(w))v))u - \rho(\alpha(u))\rho(\alpha(v))w + \rho(\alpha(u))\rho(\alpha(w))v.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&[[u, v]_\alpha, w]_\alpha + [[w, u]_\alpha, v]_\alpha + [[v, w]_\alpha, u]_\alpha\\
= \;& \rho([\alpha(v), \alpha(u)] - \alpha(\rho(\alpha(v))u - \rho(\alpha(u))v))w + \text{cycl}, \quad \forall u, v, w \in V.
\end{aligned}$$

∎

The following result can be obtained from [14, 15] in terms of the cocycle conditions. In order to be self-contained, we give a separate proof.

Theorem 3.6. ([14, 15]) *Let $\mathfrak{g}$ be a Lie algebra with finite* **k***-dimension, $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be a finite-dimensional representation of $\mathfrak{g}$ and $\alpha : V \to \mathfrak{g}$ be a linear operator. Using the same notations as in Eq. (25), $\tilde{\tilde{\alpha}}_- \in \tilde{\mathfrak{g}} \otimes \tilde{\mathfrak{g}}$ is a skew-symmetric solution of GCYBE (28) if and only if α satisfies Eq. (32) and*

$$[x, B_\alpha(u, v)] = B_\alpha(\rho(x)u, v) + B_\alpha(u, \rho(x)v) \quad \forall u, v \in V, x \in \mathfrak{g}, \tag{33}$$

where

$$B_\alpha(u, v) = [\alpha(u), \alpha(v)] - \alpha(\rho(\alpha(u))v - \rho(\alpha(v))u), \quad \forall u, v \in V. \tag{34}$$

If a linear operator α satisfies Eq. (33), it is called a **generalized $\mathcal{O}$-operator**.

One can consider more general Lie brackets and cocycle conditions than given in Eq. (31) and (34), by involving (nonzero) Lie structures on the representation spaces and Rota-Baxter operators or, more generally, $\mathcal{O}$-operators of nonzero weights. We refer the reader to [4] for some results in this direction. It would be interesting to consider explicit cycle conditions corresponding to Rota-Baxter operators with nonzero weights.

Proof. By Corollary 3.4 and Lemma 3.5 we know that $\tilde{\tilde{\alpha}}_- \in \tilde{\mathfrak{g}} \otimes \tilde{\mathfrak{g}}$ is a skew-symmetric solution of GCYBE (28) if and only if for any $u, v, w \in V$ and $a^*, b^*, c^* \in \mathfrak{g}^*$, the following equation holds

$$\begin{aligned}
&\mathrm{ad}^*_{\tilde{\mathfrak{g}}}([\tilde{\alpha}_-(u + a^*), \tilde{\alpha}_-(v + b^*)] - \tilde{\alpha}_-(\mathrm{ad}^*_{\tilde{\mathfrak{g}}}(\tilde{\alpha}_-(u + a^*))(v + b^*) - \mathrm{ad}^*_{\tilde{\mathfrak{g}}}(\tilde{\alpha}_-(v + b^*))(u + a^*)))(w + c^*)\\
&+\mathrm{ad}^*_{\tilde{\mathfrak{g}}}([\tilde{\alpha}_-(v + b^*), \tilde{\alpha}_-(w + c^*)] - \tilde{\alpha}_-(\mathrm{ad}^*_{\tilde{\mathfrak{g}}}(\tilde{\alpha}_-(v + b^*))(w + c^*) - \mathrm{ad}^*_{\tilde{\mathfrak{g}}}(\tilde{\alpha}_-(w + c^*))(v + b^*)))(u + a^*)\\
&+\mathrm{ad}^*_{\tilde{\mathfrak{g}}}([\tilde{\alpha}_-(w + c^*), \tilde{\alpha}_-(u + a^*)] - \tilde{\alpha}_-(\mathrm{ad}^*_{\tilde{\mathfrak{g}}}(\tilde{\alpha}_-(w + c^*))(u + a^*) - \mathrm{ad}^*_{\tilde{\mathfrak{g}}}(\tilde{\alpha}_-(u + a^*))(w + c^*)))(v + b^*) = 0.
\end{aligned}$$

By the proof of Theorem 2.15, this is equivalent to

$$\begin{aligned}
&\mathrm{ad}^*_{\tilde{\mathfrak g}}([\alpha(u),\alpha(v)]-\alpha(\rho(\alpha(u))v)+\alpha(\rho(\alpha(v))u)-\rho^*(\alpha(u))\alpha^*(b^*)+\alpha^*(\mathrm{ad}^*(\alpha(u))b^*)\\
&+\alpha^*(\mathrm{ad}^*_{\tilde{\mathfrak g}}(\alpha^*(b^*))u)+\rho^*(\alpha(v))\alpha^*(a^*)-\alpha^*(\mathrm{ad}^*_{\tilde{\mathfrak g}}(\alpha^*(a^*))v)-\alpha^*(\mathrm{ad}^*(\alpha(v))a^*))(w+c^*)\\
&+\mathrm{ad}^*_{\tilde{\mathfrak g}}([\alpha(v),\alpha(w)]-\alpha(\rho(\alpha(v))w)+\alpha(\rho(\alpha(w))v)-\rho^*(\alpha(v))\alpha^*(c^*)+\alpha^*(\mathrm{ad}^*(\alpha(v))c^*)\\
&+\alpha^*(\mathrm{ad}^*_{\tilde{\mathfrak g}}(\alpha^*(c^*))v)+\rho^*(\alpha(w))\alpha^*(b^*)-\alpha^*(\mathrm{ad}^*_{\tilde{\mathfrak g}}(\alpha^*(b^*))w)-\alpha^*(\mathrm{ad}^*(\alpha(w))b^*))(u+a^*)\\
&+\mathrm{ad}^*_{\tilde{\mathfrak g}}([\alpha(w),\alpha(u)]-\alpha(\rho(\alpha(w))u)+\alpha(\rho(\alpha(u))w)-\rho^*(\alpha(w))\alpha^*(a^*)+\alpha^*(\mathrm{ad}^*(\alpha(w))a^*)\\
&+\alpha^*(\mathrm{ad}^*_{\tilde{\mathfrak g}}(\alpha^*(a^*))w)+\rho^*(\alpha(u))\alpha^*(c^*)-\alpha^*(\mathrm{ad}^*_{\tilde{\mathfrak g}}(\alpha^*(c^*))u)-\alpha^*(\mathrm{ad}^*(\alpha(u))c^*))(v+b^*)=0.
\end{aligned}$$

Since for any $s^*\in V^*$ and $a^*\in\mathfrak g^*$, we have $\mathrm{ad}^*_{\tilde{\mathfrak g}}(s^*)a^*=0$, the above equation is equivalent to the following equations:

$$\mathrm{ad}^*_{\tilde{\mathfrak g}}([\alpha(u),\alpha(v)]-\alpha(\rho(\alpha(u))v-\rho(\alpha(v))u))w+\text{cycl.}=0, \tag{35}$$

$$\begin{aligned}
&\mathrm{ad}^*_{\tilde{\mathfrak g}}(-\rho^*(\alpha(u))\alpha^*(b^*)+\alpha^*(\mathrm{ad}^*(\alpha(u))b^*)+\alpha^*(\mathrm{ad}^*_{\tilde{\mathfrak g}}(\alpha^*(b^*))u))w+\mathrm{ad}^*_{\tilde{\mathfrak g}}(\rho^*(\alpha(w))\alpha^*(b^*)\\
&-\alpha^*(\mathrm{ad}^*_{\tilde{\mathfrak g}}(\alpha^*(b^*))w)-\alpha^*(\mathrm{ad}(\alpha(w))b^*))u+\mathrm{ad}^*_{\tilde{\mathfrak g}}([\alpha(w),\alpha(u)]-\alpha(\rho(\alpha(w))u-\rho(\alpha(u))w))b^*=0,
\end{aligned}\tag{36}$$

$$\begin{aligned}
&\mathrm{ad}^*_{\tilde{\mathfrak g}}(\rho^*(\alpha(v))\alpha^*(a^*)-\alpha^*(\mathrm{ad}^*_{\tilde{\mathfrak g}}(\alpha^*(a^*))v)-\alpha^*(\mathrm{ad}^*(\alpha(v))a^*))w+\mathrm{ad}^*_{\tilde{\mathfrak g}}([\alpha(v),\alpha(w)]\\
&-\alpha(\rho(\alpha(v))w)+\alpha(\rho(\alpha(w))v))a^*+\mathrm{ad}^*_{\tilde{\mathfrak g}}(-\rho^*(\alpha(w))\alpha^*(a^*)+\alpha^*(\mathrm{ad}^*(\alpha(w))a^*)\\
&+\alpha^*(\mathrm{ad}^*_{\tilde{\mathfrak g}}(\alpha^*(a^*))w))v=0,
\end{aligned}\tag{37}$$

$$\begin{aligned}
&\mathrm{ad}^*_{\tilde{\mathfrak g}}([\alpha(u),\alpha(v)]-\alpha(\rho(\alpha(u))v)+\alpha(\rho(\alpha(v))u))c^*+\mathrm{ad}^*_{\tilde{\mathfrak g}}(-\rho^*(\alpha(v))\alpha^*(c^*)+\alpha^*(\mathrm{ad}^*(\alpha(v))c^*)\\
&+\alpha^*(\mathrm{ad}^*_{\tilde{\mathfrak g}}(\alpha^*(c^*))v))u+\mathrm{ad}^*_{\tilde{\mathfrak g}}(\rho^*(\alpha(u))\alpha^*(c^*)-\alpha^*(\mathrm{ad}^*_{\tilde{\mathfrak g}}(\alpha^*(c^*))u)-\alpha^*(\mathrm{ad}^*(\alpha(u))c^*))v=0.
\end{aligned}\tag{38}$$

We shall prove

(a) Eq. (35) ⇔ Eq. (32),
(b) Eq. (36) ⇔ Eq. (37) ⇔ Eq. (38) ⇔ Eq. (33).

The proofs of these statements are similar. So we just prove that Eq. (36) holds if and only if Eq. (33) holds. Let LHS denotes the left-hand side of Eq. (36). For any $v^*,s^*\in V^*$, $w\in V$ and $x,y\in\mathfrak g$, we have $\langle \mathrm{ad}^*_{\tilde{\mathfrak g}}(v^*)w,s^*\rangle=0, \langle \mathrm{ad}^*_{\tilde{\mathfrak g}}([x,y])b^*,s^*\rangle=0$. Thus we obtain $\langle LHS,s^*\rangle=0$.

Further, for any $x\in\mathfrak g$,

$$\begin{aligned}
\langle LHS,x\rangle &= \langle w,[\rho^*(\alpha(u))\alpha^*(b^*)-\alpha^*(\mathrm{ad}^*(\alpha(u))b^*)-\alpha^*(\mathrm{ad}^*_{\tilde{\mathfrak g}}(\alpha^*(b^*))u),x]\rangle\\
&\quad+\langle u,[-\rho^*(\alpha(w))\alpha^*(b^*)+\alpha^*(\mathrm{ad}^*_{\tilde{\mathfrak g}}(\alpha^*(b^*))w)+\alpha^*(\mathrm{ad}^*(\alpha(w))b^*),x]\rangle\\
&\quad+\langle b^*,[x,[\alpha(w),\alpha(u)]-\alpha(\rho(\alpha(w))u-\rho(\alpha(u))w)]\rangle\\
&= \langle -\alpha(\rho(\alpha(u))\rho(x)w)+[\alpha(u),\alpha(\rho(x)w)],b^*\rangle-\langle[\alpha(\rho(x)w),\alpha^*(b^*)],u\rangle\\
&\quad+\langle \alpha(\rho(\alpha(w))\rho(x)u)+[\alpha(\rho(x)u),\alpha(w)],b^*\rangle+\langle[\alpha(\rho(x)u),\alpha^*(b^*)],w\rangle\\
&\quad+\langle b^*,[x,[\alpha(w),\alpha(u)]-\alpha(\rho(\alpha(w))u-\rho(\alpha(u))w)]\rangle\\
&= \langle[\alpha(u),\alpha(\rho(x)w)]-\alpha(\rho(\alpha(u))\rho(x)w)+\alpha(\rho(\alpha(\rho(x)w))u),b^*\rangle\\
&\quad+\langle[\alpha(\rho(x)u),\alpha(w)]+\alpha(\rho(\alpha(w))\rho(x)u)-\alpha(\rho(\alpha(\rho(x)u))w),b^*\rangle\\
&\quad+\langle[x,[\alpha(w),\alpha(u)]-\alpha(\rho(\alpha(w))u-\rho(\alpha(u))w)],b^*\rangle.
\end{aligned}$$

So Eq. (36) holds if and only if Eq. (33) holds. □

Remark 3.7. With the notations as above, it is in fact straightforward to show that the bracket

$$[u,v]_\alpha:=\rho(\alpha(u))v-\rho(\alpha(v))u,\quad \forall u,v\in V, \tag{39}$$

defines a Lie algebra structure on V if and only if Eq. (32) hold.

Corollary 3.8. *Let $\mathfrak{g}$ be a Lie algebra with finite* **k***-dimension.*

(a) *Let $(\mathfrak{k},\pi)$ be a $\mathfrak{g}$-Lie algebra with finite* **k***-dimension. Let $\alpha,\beta:\mathfrak{k}\to\mathfrak{g}$ be two linear maps such that α is an extended $\mathcal{O}$-operator of weight λ with extension β of mass (κ,μ). Then $\tilde{\tilde{\alpha}}_-\in(\mathfrak{g}\ltimes_{\pi^*}\mathfrak{k}^*)\otimes(\mathfrak{g}\ltimes_{\pi^*}\mathfrak{k}^*)$ is a skew-symmetric solution of GCYBE if and only if the following equations hold:*

$$\lambda\pi(\alpha([u,v]_{\mathfrak{k}}))w+\lambda\pi(\alpha([w,u]_{\mathfrak{k}}))v+\lambda\pi(\alpha([v,w]_{\mathfrak{k}}))u=0, \tag{40}$$

$$\lambda[x,\alpha([u,v]_{\mathfrak{k}})]_{\mathfrak{g}}=\lambda\alpha([\pi(x)u,v]_{\mathfrak{k}})+\lambda\alpha([u,\pi(x)v]_{\mathfrak{k}}),\quad \forall x\in\mathfrak{g},u,v,w\in\mathfrak{k}. \tag{41}$$

In particular, if $\lambda=0$, i.e., α is an extended $\mathcal{O}$-operator of weight 0 with extension β of mass (κ,μ), then $\tilde{\tilde{\alpha}}_-\in(\mathfrak{g}\ltimes_{\pi^*}\mathfrak{k}^*)\otimes(\mathfrak{g}\ltimes_{\pi^*}\mathfrak{k}^*)$ is a skew-symmetric solution of GCYBE.

(b) *Let $(\mathfrak{k},\pi)$ be a $\mathfrak{g}$-Lie algebra with finite* **k***-dimension. Let $\alpha:\mathfrak{k}\to\mathfrak{g}$ an $\mathcal{O}$-operator of weight λ. Then $\tilde{\tilde{\alpha}}_-\in(\mathfrak{g}\ltimes_{\pi^*}\mathfrak{k}^*)\otimes(\mathfrak{g}\ltimes_{\pi^*}\mathfrak{k}^*)$ is a skew-symmetric solution of GCYBE if and only if Eq.* (40) *and Eq.* (41) *hold.*

(c) *Let $\rho:\mathfrak{g}\to\mathfrak{gl}(V)$ be a finite dimensional representation of $\mathfrak{g}$. Let $\alpha,\beta:\mathfrak{k}\to\mathfrak{g}$ be two linear maps such that α is an extended $\mathcal{O}$-operator with extension β of mass κ. Then $\tilde{\tilde{\alpha}}_-\in(\mathfrak{g}\ltimes_{\rho^*}V^*)\otimes(\mathfrak{g}\ltimes_{\rho^*}V^*)$ is a skew-symmetric solution of GCYBE.*

Proof. (a) Since α is an extended $\mathcal{O}$-operator of weight λ with extension β of mass (κ,μ), for any $u,v\in\mathfrak{k}$, we have

$$B_\alpha(u,v)=[\alpha(u),\alpha(v)]_{\mathfrak{g}}-\alpha(\pi(\alpha(u))v-\pi(\alpha(v))u)=\lambda\alpha([u,v]_{\mathfrak{k}})+\kappa[\beta(u),\beta(v)]_{\mathfrak{g}}+\mu\beta([u,v]_{\mathfrak{k}}).$$

Thus, by Theorem 3.6, $\tilde{\tilde{\alpha}}_-\in(\mathfrak{g}\ltimes_{\pi^*}\mathfrak{k}^*)\otimes(\mathfrak{g}\ltimes_{\pi^*}\mathfrak{k}^*)$ is a skew-symmetric solution of GCYBE if and only if the following equations hold:

$$\pi(\lambda\alpha([u,v]_{\mathfrak{k}})+\kappa[\beta(u),\beta(v)]_{\mathfrak{g}}+\mu\beta([u,v]_{\mathfrak{k}}))w+\text{cycl.}=0, \tag{42}$$

$$\begin{aligned}&[x,\lambda\alpha([u,v]_{\mathfrak{k}})+\kappa[\beta(u),\beta(v)]_{\mathfrak{g}}+\mu\beta([u,v]_{\mathfrak{k}})]_{\mathfrak{g}}\\ =\;&\lambda\alpha([\pi(x)u,v]_{\mathfrak{k}})+\kappa[\beta(\pi(x)u),\beta(v)]_{\mathfrak{g}}+\mu\beta([\pi(x)u,v]_{\mathfrak{k}})\\ &+\lambda\alpha([u,\pi(x)v]_{\mathfrak{k}})+\kappa[\beta(u),\beta(\pi(x)v)]_{\mathfrak{g}}+\mu\beta([u,\pi(x)v]_{\mathfrak{k}}),\quad\forall x\in\mathfrak{g},u,v\in\mathfrak{k}.\end{aligned} \tag{43}$$

On the other hand, for any $u,v,w\in\mathfrak{k}$, we have

$$\begin{aligned}\kappa\pi([\beta(w),\beta(u)]_{\mathfrak{g}})v+\kappa\pi([\beta(v),\beta(w)]_{\mathfrak{g}})u&=\kappa\pi(\beta(\pi(\beta(w))u))v+\kappa\pi(\beta(\pi(\beta(v))w))u\\&=-\kappa\pi(\beta(v))\pi(\beta(w))u-\kappa\pi(\beta(u))\pi(\beta(v))w\\&=\kappa\pi(\beta(v))\pi(\beta(u))w-\kappa\pi(\beta(u))\pi(\beta(v))w\\&=\kappa\pi([\beta(v),\beta(u)]_{\mathfrak{g}})w.\end{aligned}$$

Therefore, $\kappa\pi([\beta(u),\beta(v)]_{\mathfrak{g}})w+\text{cycl.}=0$. Moreover,

$$\begin{aligned}\mu\pi(\beta([u,v]_{\mathfrak{k}}))w&=-\mu\pi(\beta(w))[u,v]_{\mathfrak{k}}\\&=-\mu[\pi(\beta(w))u,v]_{\mathfrak{k}}-\mu[x,\pi(\beta(w))v]_{\mathfrak{k}}\\&=-\mu\pi(\beta([w,u]_{\mathfrak{k}}))v+\mu[x,\pi(\beta(v))w]_{\mathfrak{k}}\\&=-\mu\pi(\beta([w,u]_{\mathfrak{k}}))v-\mu\pi(\beta([v,w]_{\mathfrak{k}}))u.\end{aligned}$$

Therefore $\mu\pi(\beta([u,v]_{\mathfrak{k}}))w+\mu\pi(\beta([w,u]_{\mathfrak{k}}))v+\mu\pi(\beta([v,w]_{\mathfrak{k}}))u=0$. So Eq. (40) holds if and only if Eq. (42) holds. Furthermore, for any $x\in\mathfrak{g}$, we have that

$$[x,\kappa[\beta(u),\beta(v)]_{\mathfrak{g}}]_{\mathfrak{g}}=\kappa[[x,\beta(u)]_{\mathfrak{g}},\beta(v)]_{\mathfrak{g}}+\kappa[\beta(u),[x,\beta(v)]_{\mathfrak{g}}]_{\mathfrak{g}}=\kappa[\beta(\pi(x)u),\beta(v)]_{\mathfrak{g}}+\kappa[\beta(u),\beta(\pi(x)v)]_{\mathfrak{g}}$$

and

$$[x, \mu\beta([u,v]_{\mathfrak{k}})]_{\mathfrak{g}} = \mu\beta(\pi(x)[u,v]_{\mathfrak{k}}) = \mu\beta([\pi(x)u,v]_{\mathfrak{k}}) + \mu\beta([u,\pi(x)v]_{\mathfrak{k}}).$$

Therefore, Eq. (41) holds if and only if Eq. (43) holds. In conclusion, $\tilde{\tilde{\alpha}}_- \in (\mathfrak{g} \ltimes_{\pi^*} \mathfrak{k}^*) \otimes (\mathfrak{g} \ltimes_{\pi^*} \mathfrak{k}^*)$ is a skew-symmetric solution of GCYBE if and only if Eq. (40) and Eq. (41) hold.

(b) This follows from Item (a) by setting $\kappa = \mu = 0$.

(c) This follows from Item (a) by setting $(\mathfrak{k}, \pi) = (V, \rho)$. □

4. Rota-Baxter operators, $\mathcal{O}$-operators and relative differential operators

In this section, we show that an $\mathcal{O}$-operator can be recovered from a Rota-Baxter operator on a large space. We also introduce a differential variation of the $\mathcal{O}$-operator and study its relation with $\mathcal{O}$-operators.

4.1. Rota-Baxter operators and $\mathcal{O}$-operators.

We start with the relationship between $\mathcal{O}$-operators on a $\mathfrak{g}$-Lie algebra and Rota-Baxter operators.

Proposition 4.1. *Let $\mathfrak{g}$ be a Lie algebra and $(\mathfrak{k}, \pi)$ be a $\mathfrak{g}$-Lie algebra. Let $\alpha : \mathfrak{k} \to \mathfrak{g}$ be a linear map and let $\lambda \in \mathbf{k}$. Then the following statements are equivalent.*

(a) *The linear map α is an $\mathcal{O}$-operator of weight λ.*

(b) *The linear map*

$$\overline{\alpha} : \mathfrak{g} \ltimes_\pi \mathfrak{k} \to \mathfrak{g} \ltimes_\pi \mathfrak{k}, \quad \overline{\alpha}(x,u) = (\alpha(u) - \lambda x, 0), \quad \forall x \in \mathfrak{g}, u \in \mathfrak{k}, \tag{44}$$

is a Rota-Baxter operator of weight λ.

(c) *The linear map*

$$-\lambda \mathrm{id} - \overline{\alpha} : \mathfrak{g} \ltimes_\pi \mathfrak{k} \to \mathfrak{g} \ltimes_\pi \mathfrak{k}, \quad (-\lambda\mathrm{id} - \overline{\alpha})(x,u) = (-\alpha(u), -\lambda u), \quad \forall x \in \mathfrak{g}, u \in \mathfrak{k}, \tag{45}$$

is a Rota-Baxter operator of weight λ.

Proof. (a)⇔(b). Let $x, y \in \mathfrak{g}, u, v \in \mathfrak{k}$. Then we have

$$\begin{aligned}
[\overline{\alpha}(x,u), \overline{\alpha}(y,v)] &= (\lambda^2[x,y] - \lambda[x,\alpha(v)] - \lambda[\alpha(u),y] + [\alpha(u),\alpha(v)], 0),\\
\overline{\alpha}([\overline{\alpha}(x,u),(y,v)]) &= (\lambda^2[x,y] - \lambda[\alpha(u),y] - \lambda\alpha(\pi(x)v) + \alpha(\pi(\alpha(u))v), 0),\\
\overline{\alpha}([(x,u),\overline{\alpha}(y,v)]) &= (\lambda^2[x,y] - \lambda[x,\alpha(v)] + \lambda\alpha(\pi(y)u) - \alpha(\pi(\alpha(v))u), 0),\\
\lambda\overline{\alpha}([(x,u),(y,v)]) &= (-\lambda^2[x,y] + \lambda\alpha(\pi(x)v) - \lambda\alpha(\pi(y)u) + \lambda\alpha([u,v]), 0).
\end{aligned}$$

Therefore

$$[\alpha(u), \alpha(v)] = \alpha(\pi(\alpha(u))v - \pi(\alpha(v))u) + \lambda\alpha([u,v])$$

if and only if

$$[\overline{\alpha}(x,u), \overline{\alpha}(y,v)] = \overline{\alpha}([\overline{\alpha}(x,u),(y,v)]) + \overline{\alpha}([(x,u),\overline{\alpha}(y,v)]) + \lambda\overline{\alpha}([(x,u),(y,v)]),$$

for any $x, y \in \mathfrak{g}, u, v \in \mathfrak{k}$.

(b)⇔(c). This follows from the following basic fact on Rota-Baxter operators: a linear map P on a Lie algebra is a Rota-Baxter operator of weight $\lambda \in \mathbf{k}$ if and only if $-\lambda\mathrm{id} - P$ is a Rota-Baxter operator of weight $\lambda \in \mathbf{k}$. □

For $\mathcal{O}$-operators on a $\mathfrak{g}$-module, we have

Corollary 4.2. *Let $\mathfrak{g}$ be a Lie algebra and (V, ρ) be a $\mathfrak{g}$-module. Let $\alpha : V \to \mathfrak{g}$ be a linear map. Let λ and $\mu \neq 0$ be in $\mathbf{k}$. The following statements are equivalent.*

(a) *The linear map* α *is an* $\mathcal{O}$*-operator (of weight 0).*

(b) *The linear map*

$$\bar{\alpha}: \mathfrak{g} \ltimes_\rho V \to \mathfrak{g} \ltimes_\rho V, \quad \bar{\alpha}(x, u) = (\mu\alpha(u) - \lambda x, 0), \quad \forall x \in \mathfrak{g}, u \in V, \tag{46}$$

is a Rota-Baxter operator of weight λ.

(c) *The linear map*

$$-\lambda \mathrm{id} - \bar{\alpha}: \mathfrak{g} \ltimes_\rho V \to \mathfrak{g} \ltimes_\rho V, \quad (-\lambda\mathrm{id} - \bar{\alpha})(x, u) = (-\mu\alpha(u), -\lambda u), \quad \forall x \in \mathfrak{g}, u \in V, \tag{47}$$

is a Rota-Baxter operator of weight λ.

Proof. Since a $\mathfrak{g}$-module (V, ρ) is a $\mathfrak{g}$-Lie algebra when V is equipped with the zero bracket, the corollary follows from Proposition 4.1 and the simple fact that a linear operator $\alpha : V \to \mathfrak{g}$ is an $\mathcal{O}$-operator if and only if $\mu\alpha$ is one for $0 \neq \mu \in \mathbf{k}$. □

By a similar argument as that for Proposition 4.1, we also obtain the following relation of invertible $\mathcal{O}$-operators with Rota-Baxter operators.

Proposition 4.3. *Let* $\mathfrak{g}$ *be a Lie algebra and* (V, ρ) *be a* $\mathfrak{g}$*-module. Let* $\alpha : V \to \mathfrak{g}$ *be an invertible linear map. Let* λ, $\mu_1 \neq 0$ *and* $\mu_2 \neq \pm\lambda$ *be in* $\mathbf{k}$. *Then* α *is an* $\mathcal{O}$*-operator of weight 0 if and only if*

$$\bar{\alpha}(x, u) = \left(\mu_1\alpha(u) - \frac{\mu_2 + \lambda}{2}x, \frac{\lambda^2 - \mu_2^2}{4\mu_1}\alpha^{-1}(x) + \frac{\mu_2 - \lambda}{2}u\right), \quad \forall x \in \mathfrak{g}, u \in V, \tag{48}$$

is a Rota-Baxter operator of weight λ *on* $\mathfrak{g} \ltimes_\rho V$.

4.2. **Rota-Baxter operators and relative differential operators.** We first define a relative version of the differential operator which can also be regarded as a differential variation of the $\mathcal{O}$-operator.

Definition 4.4. Let $\mathfrak{g}$ be a Lie algebra.

(a) Let $(\mathfrak{k}, \pi)$ be a $\mathfrak{g}$-Lie algebra. A linear map $f : \mathfrak{g} \to \mathfrak{k}$ is called a **relative differential operator (on** $\mathfrak{k}$**) of weight** λ if

$$f([x, y]_\mathfrak{g}) = \pi(x)f(y) - \pi(y)f(x) + \lambda[f(x), f(y)]_\mathfrak{k}, \quad \forall x, y \in \mathfrak{g}. \tag{49}$$

(b) Let (V, ρ) be a $\mathfrak{g}$-module. A linear map $f : \mathfrak{g} \to V$ is called a **relative differential operator (on** V**)** if

$$f([x, y]_\mathfrak{g}) = \pi(x)f(y) - \pi(y)f(x), \quad \forall x, y \in \mathfrak{g}. \tag{50}$$

A relative differential operator on a $\mathfrak{g}$-module V can be regarded as a special case of a relative differential operator on a $\mathfrak{g}$-Lie algebra $\mathfrak{k}$ when V is equipped with the trivial Lie bracket.

Proposition 4.5. *Let* $\mathfrak{g}$ *be a Lie algebra and* $(\mathfrak{k}, \pi)$ *be a* $\mathfrak{g}$*-Lie algebra. Let* $f : \mathfrak{g} \to \mathfrak{k}$ *be a linear map. Then the following statement are equivalent.*

(a) *The linear map* f *is a relative differential operator of weight 1.*

(b) *The linear map*

$$\bar{f}: \mathfrak{g} \ltimes_\pi \mathfrak{k} \to \mathfrak{g} \ltimes_\pi \mathfrak{k}, \quad \bar{f}(x, u) = (x, f(x)), \quad \forall x \in \mathfrak{g}, u \in \mathfrak{k}, \tag{51}$$

is a Rota-Baxter operator of weight -1.

(c) *The linear map*

$$\mathrm{id} - \bar{f}: \mathfrak{g} \ltimes_\pi \mathfrak{k} \to \mathfrak{g} \ltimes_\pi \mathfrak{k}, \quad (\mathrm{id} - \bar{f})(x, u) = (0, u - f(x)), \quad \forall x \in \mathfrak{g}, u \in \mathfrak{k}, \tag{52}$$

is a Rota-Baxter operator of weight -1.

Proof. (a)⇔(b). Let $x, y \in \mathfrak{g}, u, v \in \mathfrak{k}$. Then we have

$$\begin{aligned}
[\bar{f}(x,u), \bar{f}(y,v)] &= ([x,y], \pi(x)f(y) - \pi(y)f(x) + [f(x), f(y)]);\\
\bar{f}([\bar{f}(x,u),(y,v)]) &= ([x,y], f([x,y]));\\
\bar{f}([(x,u), \bar{f}(y,v)]) &= ([x,y], f([x,y]));\\
\bar{f}([(x,u),(y,v)]) &= ([x,y], f([x,y])).
\end{aligned}$$

Therefore

$$[\bar{f}(x,u), \bar{f}(y,v)] = \bar{f}([\bar{f}(x,u),(y,v)]) + \bar{f}([(x,u), \bar{f}(y,v)]) - \bar{f}([(x,u),(y,v)])$$

if and only if

$$f([x,y]) = \pi(x)f(y) - \pi(y)f(x) + [f(x), f(y)].$$

for any $x, y \in \mathfrak{g}, u, v \in \mathfrak{k}$.

(b)⇔(c). This follows from the same reason as in the case of Proposition 4.1. □

By the same argument as for Corollary 4.2, we have:

Corollary 4.6. *Let $\mathfrak{g}$ be a Lie algebra and (V,ρ) be a $\mathfrak{g}$-module. Let $f : \mathfrak{g} \to V$ be a linear map. Let $\lambda, \mu \in \mathbf{k}$ be nonzero. Then the following statements are equivalent.*

(a) *The linear map f is a relative differential operator.*

(b) *The linear map*

$$\bar{f} : \mathfrak{g} \ltimes_\rho V \to \mathfrak{g} \ltimes_\rho V, \quad \bar{f}(x,u) = (-\lambda x, \mu f(x)), \quad \forall x \in \mathfrak{g}, u \in V, \tag{53}$$

is a Rota-Baxter operator of weight λ.

(c) *The linear map*

$$-\lambda \mathrm{id} - \bar{f} : \mathfrak{g} \ltimes_\rho V \to \mathfrak{g} \ltimes_\rho V, \quad (-\lambda \mathrm{id} - \bar{f})(x,u) = (0, -\mu f(x) - \lambda u), \quad \forall x \in \mathfrak{g}, u \in V, \tag{54}$$

is a Rota-Baxter operator of weight λ.

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